Foundations of Quantum Group Theory

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Introduction

This is an introduction to the foundations of quantum group theory. Quantum groups or *Hopf algebras* are an exciting new generalisation of ordinary groups. They have a rich mathematical structure and numerous roles in situations where ordinary groups are not adequate. The goal in this volume is to set out this mathematical structure by developing the basic properties of quantum groups as objects in their own right; what quantum groups are conceptually and how to work with them. We will also give some idea of the meaning of quantum groups for physics. On the other hand, just as ordinary groups have all sorts of applications in physics, not one specific application but many, in the same way one finds that quantum groups have a wide variety of probably unrelated applications. This diversity is one of the themes in the volume and is a good reason to focus on quantum groups as mathematical objects.

This book is not a survey; many of the most interesting recent results in representation theory, applications in conformal field theory and low-dimensional topology, etc., are not discussed in any detail. In this sense, there is less material here than in my lecture notes [1]. In place of this fashionable material, I have developed the pedagogical side of [1], giving now more details of proofs and solutions to exercises, and in general concentrating more on that part of the theory of quantum groups that can be considered as firmly established. I have also included my more recent work on braided groups.

This text is addressed primarily to theoretical physicists and mathematicians wishing to begin work on quantum groups. For physicists, I have tried to give full details and line-by-line proofs of all the basic results that are needed for research in the field. Also, I have struggled hard to maintain an informal style so that the essential content is not too obscured by inessential formalism. For mathematicians, I have adopted a theorem–proof format so that the main results can be understood clearly, and have endeavoured not to say things that are technically false. This balance between readability and rigour is achieved by taking a completely algebraic line and not discussing in any depth the equally interesting variants of quantum groups based on C^*-algebras and Hopf–von Neumann or Kac algebras. In other words, we limit ourselves primarily to the algebraic theory of quantum groups rather than to the functional-analytic theory.

What is a quantum group? To answer this question, let us first consider what is a group. There are several answers. The most familiar point of view about groups is as collections of transformations. Transformations of a space are assumed invertible, and every closed collection of invertible transformations is, inevitably, a group. This is the role of groups as symmetries. Quantum groups, too, can act on things. However, now the transformations are not all invertible. Instead, quantum groups have a weaker structure, called the antipode S, which provides a nonlocal 'linearised inverse'. It means that now not individual elements but certain linear combinations are invertible. Remarkably, this weaker invertibility is all that is actually used in applications. For example, just as groups can act on themselves in the adjoint representation (which would appear to require an inverse), so quantum groups act on themselves in an adjoint representation. Likewise, just as every representation of a group has a conjugate one (provided by the action of the inverse group element), so every quantum group representation has a conjugate provided by the antipode.

A second point of view about groups is that their representations have a tensor product. This is familiar in particle physics. For example, if J_z is an angular momentum operator, then an element $\Delta J_z = J_z \otimes 1 + 1 \otimes J_z$ provides its action on tensor products (this linear addition is characteristic of ordinary Lie algebras: quantum groups tend to be more complicated). The tensor product is symmetric, the symmetry being implemented by the usual transposition of vector spaces. Quantum group representations, too, have a tensor product. In fact, we will see a theorem that given any collection of objects which can be identified with vector spaces, compatible with the tensor product of vector spaces, we can reconstruct a quantum group and identify the collection as its representations. So, in a certain context, this is a complete characterisation. For strict quantum groups (ones possessing a so-called 'universal R-matrix'), the tensor product of representations is symmetric (just as for representations of groups), but now only up to isomorphism. This isomorphism is not given by the usual transposition but by a weaker structure called a quasisymmetry or 'braiding' Ψ. It is weaker because, in general, Ψ does not obey $\Psi^2 = \mathrm{id}$. Instead, it provides an action of the braid group rather than of the symmetric group. It is this fact that leads to the application of strict quantum groups in low-dimensional topology.

These two points of view cover the most well-known settings in which quantum groups have arisen, namely their connection with quantum inverse scattering, exactly solvable lattice models and low-dimensional topology. In this context, quantum groups arose as symmetries of quantum statistical systems, leading to braid group representations in these systems, as well as in conformal field theories related to their continuum limit. This theory will obviously take up a substantial part of the volume, namely Chapters 2–4 and parts of Chapters 7, 8 and 9. Important, but not the only, examples are the quantum groups $U_q(g)$ introduced by V.G. Drinfeld and M. Jimbo as deformations of the enveloping algebras of complex simple Lie algebras.

However, Hopf algebras in general have a further unusual property or *raison d'être* quite different from their role as generalised symmetry. This gives the third and fourth ideas about what a quantum group or Hopf algebra *is*. These are connected with their duality or self-duality properties and are the author's own reason to be interested in quantum groups. From this point of view, a Hopf algebra is an algebra for which the dual linear space of the algebra is also an algebra. The algebra structure on the dual linear space is expressed in terms of the original algebra A as a coproduct or comultiplication map $\Delta : A \to A \otimes A$. Supplementing an algebra by a comultiplication (forming a coalgebra) restores a kind of input–output symmetry to the system. When A is the algebra of observables of a classical or quantum system, then the ordinary multiplication $A \otimes A \to A$ corresponds to logical deduction (multiplication of projection operators in the quantum case, or simply multiplication of the characteristic functions in the classical case, which is intersection of their underlying sets). By contrast, Δ allows the reverse operation, to 'unmultiply' (comultiply). The comultiplication of an element X in A is the sum of all those things in $A \otimes A$ which could give X when combined according to an underlying group structure. For example, if X is the coordinate function on the real line, $\Delta X = X \otimes 1 + 1 \otimes X$ expresses linear addition on the line. The probabilistic interpretation is that X in A is a random variable, while $X_1 = X \otimes 1$ and $X_2 = 1 \otimes X$ are two independent random variables embedded in $A \otimes A$, which is the system after two steps in a random walk. Embedding X in $A \otimes A$ as $\Delta X = X_1 + X_2$ says precisely that our total position X after two steps is the sum of the two random variables X_1, X_2. This ΔX represents all the ways to obtain X after two steps. Thus, the comultiplication represents 'induction' or possibility rather than deduction. Remarkably, the rules for Δ are just the same as the rules for multiplication, with the arrows reversed. This remarkable way of understanding probability and random walks on groups was one of the classical reasons for interest in Hopf algebras some years ago. This work has naturally had a renaissance with the arrival of quantum groups *en masse*. This is the third idea of what a Hopf algebra is, and is the topic of Chapter 5.

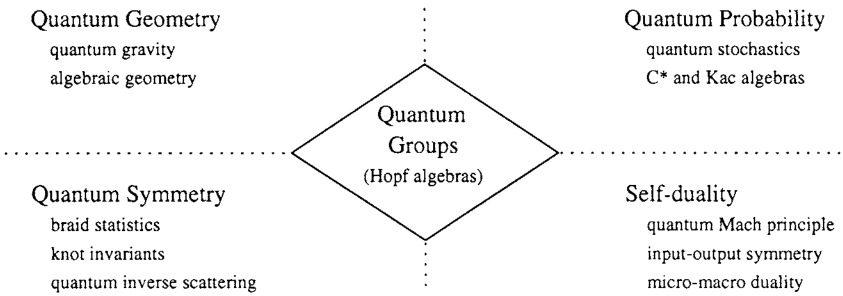
Finally, we must recall that non-Abelian Lie groups are after all the simplest examples of Riemannian geometry with curvature. It is well-known that to do Riemannian geometry on a manifold M it is often convenient to work with the algebra of functions $A = C(M)$. For example, a vector field is a derivation in such algebraic terms. The idea of noncommutative (algebraic) geometry is that even when this algebra A is made into a noncommutative one, one can continue to do geometry, perhaps even Riemannian geometry, provided all our constructions are referred to the algebra A rather than to any manifold M, which need no longer exist. This is an old idea, but one that was developed significantly in recent years by A. Connes and others. If our initial M is phase-space then when A is quantised it becomes noncommutative in just this way (with noncommutativity controlled by $\hbar$). We can still continue to think of it as like '$C(M)$', although, in truth, the points in M no longer exist because the position and momentum coordinates can no longer be measured simultaneously. We can still continue to do geometry in this setting. This is 'quantum Riemannian geometry'. In very general mathematical terms, such a point of view can also be taken in all the above contexts, notably the matrix quantum groups of function algebra type (Chapter 4), where matrix multiplication can be done in the noncommutative setting, or the context of quantum random walks. However, let us ask more specifically about quantum Riemannian geometry. According to our view of Lie groups, noncocommutative Hopf algebras (i.e. with a noncommutative comultiplication) are like non-Abelian groups, i.e. they have curvature. If, at the same time, they are noncommutative as a result of quantisation, then we have a quantum system combined in a consistent way with curvature, i.e. models of quantum-gravity. This was the author's original motivation for Hopf algebras, and is a fourth *raison d'être* for them.

Specifically, it was investigated (by the author) under what conditions the quantum algebra of observables of a particle on a homogeneous space is, in fact, a Hopf algebra. It turns out that it is sufficient for the homogeneous metric to obey a second order 'Einstein' equation. Solving this, one finds a large class of noncommutative and noncocommutative Hopf algebras related to group factorisations – quite different and independent from those of Drinfeld, Jimbo, *et al.* While not connected with braiding, they are characterised instead by a remarkable self-duality property: the dual Hopf (von Neumann) algebra is of just the same type with the roles of position and momentum interchanged. It corresponds to the quantum particle moving on a dual or 'mirror' homogeneous space. We will see this in Chapter 6.

In this setting, Hopf algebra duality takes a very concrete form as a symmetry between quantum observables and quantum states. The dual Hopf algebra is built on A^*, i.e. the algebra of observables of the dual system is the algebra of states (induced by the comultiplication Δ) in terms of the original quantum system A. If a is in A (a quantum observable) and ϕ in A^* (a quantum state) then $\phi(a)$ (the expectation value of a in state ϕ) is interpreted in the dual system as $a(\phi)$ (the expectation value of ϕ in state a from the dual point of view). From this point of view, geometry, in the form of the simplest models of curvature in phase-space, is the dual of quantisation. The noncommutativity of quantisation is mirrored in these models by the noncommutativity of covariant derivatives in Riemannian geometry, expressed Hopf algebraically as noncommutativity of Δ. Thus Hopf algebras provide a unique setting for the unification of quantum mechanics and gravity, in which they appear as the same structure but in dual form; one on the algebra of observables and the other in the algebra of states. It should be appreciated that Hopf algebras provide in this way an example of a general phenomenon: the dual relationship between quantum theory (as an outgrowth of arithmetics and intuitionistic logic) and geometry. In maintaining this duality, our notion of Riemannian geometry must itself be enlarged to the noncommutative geometric setting, i.e. the gravitational field is 'quantised', albeit not in a very conventional sense.

As a concrete demonstration of these ideas, it turns out that a differentiable quantum dynamical system of a particle on a line is a Hopf algebra of self-dual type if and only if the particle moves in something like the background of a black-hole type metric. This approach to quantum-gravity or consistent physics at the Planck scale is one of the themes of Chapter 6 from a physical point of view. Of course, models of this type are quite simple, just as Lie groups are only the simplest Riemannian manifolds. The duality which we describe can be viewed as a generalisation of usual wave–particle duality at least to such spaces, a generalisation made possible by quantum groups. It seems likely that it can be taken further and related also to other duality phenomena running throughout theoretical physics.

Probably the most remarkable thing about Hopf algebras is that any one of the four points of view above would be reason enough to invent Hopf algebras or quantum groups as a generalisation of groups. Yet the same mathematical structure serves all four simultaneously, and therefore provides the framework for some truly remarkable conceptual unifications of these four directions! The comultiplication Δ leads simultaneously to a tensor product of representations (as in particle physics), to a convolution algebra of states expressing random walks, and to quantum-geometric group structure on phase-space expressing curvature. We do not hope to discuss all the myriad applications of Hopf algebras and their generalisa-



Some physical origins of quantum groups.

tions resulting from this. Let us mention only a few important directions. First, the unification of our third and fourth points of view about quantum groups surely will suggest a better understanding of the deep connection between statistical mechanics (or thermodynamics) and gravity already suggested by Hawking radiation. Indeed, a great deal about entropy, time-reversal symmetries in logic and other ideas can be explored by means of Hopf algebra models, and connected via noncommutative geometry with gravity.

Secondly, the connections between these subjects and quantum groups in the deformation theoretic sense may lead to tools for eliminating or at least regularising the infinities in quantum gravity which occur in the conventional path-integral point of view. To explain this it should be mentioned that a lot of work in quantum groups under the heading of the first and second areas above has been a process of formal q-deformation, i.e. generalising groups of symmetry by putting in a parameter q. There are powerful existence and uniqueness results coming from the work of V.G. Drinfeld, and a general formalism of bialgebra deformations due to M. Gerstenhaber and S.D. Schack. Note that q can have any meaning depending on the application (and is not usually related to physical $\hbar$).

In particular, we can introduce q-deformation as a way to regularise elementary particle quantum field theories, with the usual infinities expressed as poles $1/(q-1)$. Any group symmetries are preserved as quantum group symmetries. Then we renormalise and set q to 1. There may be an anomaly, but the final symmetry should still be a quantum group one. Moreover, as we will see, making a q-deformation is much more systematic and less *ad hoc* than other forms of regularisation because the resulting correlation functions, etc. are simply q-deformed versions of their usual expressions, and have similar algebraic properties. For example, exponentials become q-exponentials. This is related to the fact that differential operators deform to q-difference operators. In this scheme, infinities that arise from the small-scale structure of spacetime are literally controlled by replacing differential operators by finite differences, but in a more systematic way than simply working in a discrete lattice. This is surely an important potential application of quantum group technology under the first and second headings above.

Finally, we can apply such a q-regularisation procedure to the usual approach for quantum gravity itself. The radical suggestion coming from the above is that we need not, after all, renormalise, i.e. that the parameter q can after all be identified with a function of $\hbar$. This means quite simply that the infinities in the path-integral approach to quantum gravity are a product of incorrectly using classical geometry inside the path-integral. In reality the true geometry that we must use is noncommutative geometry, i.e. already partially quantised. If we use this quantum geometry from the start then we may not run into unavoidable infinities in quantum gravity. In some sense, the unifications suggested by Hopf algebras indicate that corrections due to quantum effects to the small-scale structure of spacetime (or phase-space) are of the nature of replacing differential operators by certain kinds of finite differences, with comparably nice properties.

In this respect then, Chapters 5 and 6 are central to one set of applications of Hopf algebras to physics, namely to Planck-scale physics. Chapter 5 develops the probabilistic interpretation as explained above, while Chapter 6 summarises results from the author's Ph.D. thesis about Planck-scale physics. A number of algebraic aspects of general cross product quantisation and extension theory are introduced at the same time, which are surely useful in a wider context as well.

Chapter 7 returns to the theory of quasitriangular Hopf algebras with Drinfeld's quantum double and its generalisations. Chapter 8 gives some of the semiclassical ideas that led to the quantum groups $U_q(g)$. Chapter 9 then proceeds with the formal (category-theoretic) aspects of the representation theory of quantum groups, focusing on the braiding. Here a new idea arises, namely the role of quantum groups as generating categories within which live other algebraic structures that we know and love (and perhaps want to generalise). This is also due to the author and is developed further in Chapter 10. We will see that a certain two-dimensional quantum group $\mathbb{Z}'_{/2}$ has as its category of representations (according to our second point of view above on quantum groups) the category of super-vector spaces. But we can go further: a certain n-dimensional quantum group has as its category of representations the category of anyonic vector spaces, and so on. Moreover, we can start to generalise ideas familiar in supersymmetry to these more general settings. We learn some new things about supersymmetry itself. For example, every Hopf algebra containing a group-like element of order 2 can be turned (transmuted) into a super one. Likewise, if order n, it can be transmuted into an anyonic quantum group. In the reverse direction, every group or quantum group in such categories (such as super or anyonic) can be bosonised to an ordinary quantum group by a process of bosonisation. Thus the category in which an object lives is actually quite fluid: it can be changed like a 'change of coordinates'. In physical terms, it means that quantum and statistical noncommutativity can be interchanged. For example, we need never work with supergroups provided we do not mind working with (their bosonised) quantum groups instead. On the other hand, it might be more natural sometimes to take a quantum system and understand it as the bosonisation of something simpler. For example, the ordinary Weyl algebra of quantum mechanics (which is a certain peculiar Hopf algebra) is nothing other than the bosonisation of a braided version of the real line. Such things as q-difference operators mentioned above are nothing other than the result of bosonising the obvious braided differential operators. Thus, just as quantisation can give braided structures, braidings can give quantum ones.

We can take this point of view to its logical conclusion and view every quantum group in its braided category of representations. All of its quantum aspect now appears in the braiding Ψ in the category. What is left, the resulting 'braided group', is braided commutative and braided cocommutative, i.e. appears more like a (braided) Abelian group. This leads to a number of new results for quantum groups by thinking about them in this way. Braided groups in general also provide a new systematic approach to q-deformation starting from the (braided) addition law on q-deformed $\mathbb{R}^n$. This includes natural q-deformed Euclidean and Minkowski spaces. One of the goals of this volume is to lead up to this theory of braided groups. We will be able to cover in this final Chapter 10 only the basic definitions and ideas of this braided approach to the theory of quantum groups and q-deformation. The theory of braided groups is due to the author and collaborators.

It is hoped that the further theory of braided groups, as well as the more advanced theory of quantum groups and quantum geometry, will be developed in a sequel to the present volume. Important topics which must await this sequel are: the axiomatic theory of differential graded algebras and bicovariant differentials on quantum groups, quantum group gauge theory [2], the general theory of braided groups [3, 4] and braided-Lie algebras [5], and applications to the q-regularisation of quantum field theory. Other major omissions are the general abstract deformation theory and star products, the advanced theory of the quantum Weyl group and the canonical or crystal basis, and the advanced theory of $U_q(g)$ at roots of unity and its connections with Kac–Moody algebras and conformal field theory.

I will try to mention some of these further topics and also give the briefest of historical discussions in the 'Notes' sections at the end of each chapter. My aim in the volume, however, is to give a systematic and pedagogical development rather than a historical one. Moreover, it is inevitable that I have emphasised the points of view developed in my own research work, particularly in the later chapters. Let me apologise therefore in advance for the brevity and, no doubt, incompleteness of these Notes at the end of each chapter. A full discussion of all points of interest would surely be a text in itself.

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